The Real Schur Decomposition of HD: Revealing the Rotations Hidden in Iterated Hadamard Transforms

1. Introduction

Consider the transformation

$$M = HD,$$

where H is a normalized Walsh–Hadamard transform and D is a diagonal matrix whose entries are independently chosen from $+1$ and -1.

Both H and D are real orthogonal matrices:

$$H^T H = I, \qquad D^T D = I.$$

Consequently,

$$M^T M = (HD)^T (HD) = D^T H^T HD = I.$$

Thus HD is itself a real orthogonal transformation.

This has an important consequence for iteration. Repeated application

$$x, \quad Mx, \quad M^2 x, \quad M^3 x, \dots$$

cannot cause exponential growth or decay of the vector norm. Instead, the transformation consists of rotations and possible reflections in suitable coordinates.

The **real Schur decomposition** provides a particularly useful way to expose this structure.

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2. Why eigenvalues are not quite enough

For a real matrix, eigenvalues can be either real or complex.

If

$$Mv = \lambda v$$

and λ is complex, then the corresponding eigenvector is generally complex as well.

For a real matrix, non-real eigenvalues occur in conjugate pairs:

$$\lambda, \lambda^*.$$

For an orthogonal matrix, every eigenvalue has unit magnitude:

$$|\lambda| = 1.$$

Therefore the complex eigenvalues can be written

$$\lambda = e^{i\theta}.$$

The angle

$$\theta = \arg(\lambda)$$

is naturally interpreted as a rotation angle.

However, the eigenvector associated with $e^{i\theta}$ is complex. It is therefore not immediately obvious what this means in the original real vector space.

The real Schur decomposition resolves this elegantly.

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3. The real Schur decomposition

Every real square matrix can be written

$$\boxed{A = QTQ^T}$$

where Q is a real orthogonal matrix and T is real **quasi-upper-triangular**.

The diagonal of T consists of blocks of two possible types:

$$[\lambda]$$

for real eigenvalues, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for a complex-conjugate pair.

Thus the real Schur form does not require complex arithmetic to represent complex eigenvalues.

Instead, each conjugate pair is represented by a real 2×2 block.

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4. The meaning of a complex eigenvector

Suppose

$$Av = \lambda v$$

where

$$v = u + iw$$

and

$$\lambda = e^{i\theta}.$$

Substituting gives

$$A(u + iw) = (\cos\theta + i\sin\theta)(u + iw).$$

Equating real and imaginary parts gives

$$Au = \cos\theta\, u - \sin\theta\, w$$

and

$$Aw = \sin\theta\, u + \cos\theta\, w.$$

Therefore the real vectors u and w span an invariant two-dimensional subspace.

Within this plane, A acts as an ordinary rotation:

$$\begin{pmatrix} u' & w' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u & w \end{pmatrix}.$$

This is the key geometric interpretation.

A complex eigenvalue pair of a real matrix corresponds to an ordinary real two-dimensional rotation plane.

The complex representation is simply a compact way of describing that real rotation.

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5. What makes HD special?

For

$$M = HD,$$

both factors are orthogonal, so M is orthogonal.

Consequently, its Schur form has a particularly simple structure.

Ignoring possible $+1$ and -1 eigenvalues, it can be represented as a collection of 2×2 rotation blocks:

$$T = \begin{pmatrix} R(\theta_1) & & & \\ & R(\theta_2) & & \\ & & \ddots & \\ & & & R(\theta_k) \end{pmatrix},$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus, in the appropriate coordinate system,

$$HD$$

is simply a collection of independent rotations, together with possible one-dimensional reflections.

This gives a very concrete interpretation of iterated HD.

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6. Iteration becomes especially simple

Suppose one of the Schur blocks is

$$R(\theta).$$

Then

$$R(\theta)^n = R(n\theta).$$

Consequently, after n applications of HD, that component of the vector has rotated through

$$n\theta.$$

Equivalently, the associated complex eigenvalue evolves as

$$\lambda^n = (e^{i\theta})^n = e^{in\theta}.$$

Thus the eigenvalue angle is the natural oscillation frequency of that mode under repeated application of HD.

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7. How is the Schur decomposition computed?

The standard numerical route is based on the **QR algorithm**.

First perform a QR decomposition:

$$A = Q_0 R_0.$$

Then reverse the factors:

$$A_1 = R_0 Q_0.$$

Because

$$A_1 = Q_0^T A Q_0,$$

A_1 is orthogonally similar to A, so it has exactly the same eigenvalues.

The process is repeated:

$$A_k = Q_k R_k,$$

$$A_{k+1} = R_k Q_k.$$

With suitable shifts and numerical refinements, this process converges toward the real Schur form.

Practical eigensolvers usually first reduce the matrix to Hessenberg form using Householder transformations, then apply a shifted QR algorithm.

Conceptually, however, the essential operation is:

$$\boxed{A \to QR \to RQ \to \text{repeat} \to T}$$

while accumulating the orthogonal transformations.

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8. Why QR iteration preserves the relevant geometry

Each QR iteration replaces

$$A_k$$

with

$$A_{k+1} = Q_k^T A_k Q_k.$$

This is an orthogonal change of coordinates.

It therefore does not change the eigenvalues, and it preserves the underlying geometric transformation.

The algorithm is progressively finding a coordinate system in which the transformation becomes simpler.

At convergence,

$$T = Q^T A Q.$$

The matrix T is therefore the same linear transformation viewed in a specially chosen orthogonal coordinate system.

For an orthogonal HD, that coordinate system exposes its independent rotation planes.

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9. Connection with complex eigenvalues

Suppose the Schur form contains

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^2 - 2\cos\theta\,\lambda + 1.$$

Its roots are

$$\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

Thus the following three descriptions are equivalent:

$$\boxed{R(\theta)}$$

$$\boxed{e^{\pm i\theta}}$$

and

$$\boxed{\text{a real two-dimensional rotation through } \theta.}$$

The complex eigenvalues are therefore not an additional mysterious structure. They are another representation of the real rotation blocks.

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10. A useful experiment with HD

This gives a straightforward experimental program.

Choose a dimension N, construct a normalized Hadamard matrix H, and generate a random diagonal matrix

$$D = \mathrm{diag}(d_1, \ldots, d_N), \qquad d_i \in \{-1, +1\}.$$

Construct

$$M = HD.$$

Then compute the real Schur decomposition

$$M = QTQ^T.$$

Inspect the diagonal blocks of T.

For every 2×2 rotation block

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the corresponding complex eigenvalues can be calculated from its characteristic polynomial.

For an exactly orthogonal rotation block, the angle can simply be obtained from

$$\theta = \mathrm{atan2}(c, a)$$

after putting the block into the standard rotation orientation.

Alternatively, calculate the complex eigenvalue

$$\lambda = a + ib$$

and use

$$\theta = \operatorname{atan2}(\operatorname{Im}\lambda, \operatorname{Re}\lambda).$$

The resulting collection

$$\theta_1, \theta_2, \ldots, \theta_k$$

is the **rotation spectrum** of the particular HD transformation.

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11. Visualising the spectrum

There are two particularly useful visualisations.

Eigenvalues in the complex plane

Plot each eigenvalue

$$\lambda = a + ib$$

as a point (a, b).

For an orthogonal matrix, the points should lie on the unit circle:

$$a^2 + b^2 = 1.$$

The angular position of each point gives the rotation angle.

Rotation-angle histogram

Alternatively, plot

$$\theta = \operatorname{atan2}(b, a)$$

for every eigenvalue.

This gives a direct view of the oscillatory modes present in the iterated transformation.

Different random choices of D can then be compared to see how the rotation spectrum changes.

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12. Why this may be interesting for iterated HD

The usual description of a Hadamard transform emphasizes its fast mixing of coordinates. Introducing the random sign matrix D adds another orthogonal operation.

But when

$$HD$$

is repeatedly applied,

$$(HD)^2, \quad (HD)^3, \quad (HD)^4, \ldots$$

the transformation is not simply "randomly scrambling" the vector at each iteration.

It is one fixed orthogonal transformation being repeatedly applied.

The Schur decomposition tells us that, in a suitable coordinate system, this repeated operation is equivalent to simultaneously rotating several independent two-dimensional components at different angular rates.

Thus an apparently complicated high-dimensional iteration can be understood as a collection of simple oscillators.

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13. An important question: what determines the rotation angles?

This is where the experiment becomes more interesting.

The Hadamard transform has a very strong algebraic structure:

$$H^2 = I.$$

The random sign matrix also satisfies

$$D^2 = I.$$

Both are involutions. Their product

$$HD$$

is therefore a product of two involutions.

In ordinary two-dimensional geometry, the product of two reflections is a rotation, with the rotation angle related to the angle between their reflecting subspaces.

The high-dimensional HD construction is a generalisation of this phenomenon.

The resulting Schur rotation angles are therefore determined by the relative geometry of the two involutions H and D.

This suggests an interesting research question:

> **What statistical structure does the Hadamard geometry impose on the rotation spectrum of HD when D is a random sign diagonal?**

The real Schur decomposition provides a direct computational tool for investigating that question.

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14. Summary

For a real matrix A,

$$A = QTQ^T$$

expresses the transformation in a particularly useful real coordinate system.

For a general matrix, T contains 1×1 and 2×2 blocks.

For the orthogonal matrix

$$M = HD,$$

the important 2×2 blocks are rotations:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Their complex eigenvalues are

$$e^{\pm i\theta}.$$

Therefore:

$$\boxed{\text{real Schur block} \longleftrightarrow \text{complex eigenvalue pair} \longleftrightarrow \text{rotation plane.}}$$

This gives a useful conceptual bridge between numerical linear algebra and the behaviour of iterated HD: **the apparently complicated high-dimensional dynamics can be decomposed into independent rotational modes, each with its own characteristic angle and hence its own oscillation frequency.**